Approximately Injectivity for Banach modules
Amin Mahmoodi

Department of Mathematics, Central Tehran Branch, Islamic Azad University, Tehran, Iran

Abstract

We introduce the notion of approximately injectivity for Banach modules. For a locally compact group $G$, we study the approximately injectivity of Banach $L^1(G)$-modules.

Keywords: injective module, approximately injective module, approximately amenability.

© 2011 Published by Islamic Azad University-Karaj Branch.

1 Introduction

The concept of amenability for Banach algebras was introduced by B. E. Johnson in 1972 [J]. Several modifications of this notion were introduced by F. Ghahramani and R. J. Loy in [GhL]. Let $A$ be a Banach algebra, and let $E$ be a Banach $A$-bimodule. A derivation $D: A \rightarrow E$ is a bounded linear map satisfying $D(ab) = D(a) \cdot b + a \cdot D(b)$, for $a, b \in A$. A derivation $D: A \rightarrow E$ is approximately inner if there exists a net $(x_\alpha)_{\alpha} \subseteq E$ such that $D(a) = \lim_{\alpha} (a \cdot x_\alpha - x_\alpha \cdot a)$, for every $a \in A$. A Banach algebra $A$ is approximately amenable if for any $A$-bimodule $E$, every derivation $D: A \rightarrow E^*$ is approximately inner. One can see [GhL] and [GhLZ] for more details.

Let $A$ be a Banach algebra with unitization $A^\sharp$. We consider unconditional unitization, i.e. if $A$ already has an identity, we adjoint another one. Let $E$ and $F$ be Banach spaces. We denote by $\mathcal{L}(E, F)$, the Banach space of all bounded linear operators from

\[1\] Corresponding Author. E-mail Address: a_mahmoodi@iauctb.ac.ir
$E$ to $F$, and write $\mathcal{L}(E)$ for $\mathcal{L}(E,E)$, and write $id_E$ for the identity map on $E$. Let $E$ and $F$ be left Banach $\mathcal{A}$-modules. Then $\mathcal{A}\mathcal{L}(E,F)$ denotes the left $\mathcal{A}$-module homomorphism in $\mathcal{L}(E,E)$. The projective tensor product $\mathcal{A}^\sharp \hat{\otimes} E$ becomes a left Banach $\mathcal{A}$-module through

$$ a \cdot (b \otimes x) := ab \otimes x \quad (a \in \mathcal{A}, \ b \in \mathcal{A}^\sharp, \ x \in E). $$

For the definitions of projective and injective modules, see [H] or [D].

The main references for this paper are [DP] and [R, Chapter5]. In section one, among other things, we introduce the notion of approximately injectivity. In section two, we study the relation between approximately amenability of the Banach algebra $L^1(G)$, for a locally compact group $G$, and approximately injectivity of Banach $L^1(G)$-modules.

## 2 Approximately injectivity

Let $\mathcal{A}$ be a Banach algebra and let $E$ be a left Banach $\mathcal{A}$-module. The module action extends to $\mathcal{A}^\sharp$ by letting

$$ (a + \lambda e_{\mathcal{A}^\sharp}) \cdot x := a \cdot x + \lambda x \quad (a \in \mathcal{A}, \ x \in E, \ \lambda \in \mathbb{C}), $$

where $e_{\mathcal{A}^\sharp}$ denotes the identity element of $\mathcal{A}^\sharp$. Therefore, the left $\mathcal{A}$-module homomorphism

$$ \Pi_{\mathcal{A}^\sharp,E} : \mathcal{A}^\sharp \hat{\otimes} E \longrightarrow E, \quad a \otimes x \longmapsto a \cdot x, $$

is well-defined.

Let $F$ be a closed subspace of a Banach space $E$. A projection from $E$ onto $F$ is an element $P \in \mathcal{L}(E)$ with $P^2 = P$ and $P(E) = F$. In this case, we say $F$ is complemented in $E$.

Let $E$ and $F$ be Banach spaces, and let $T \in \mathcal{L}(E,F)$. Then $T$ is admissible if $\ker T$, the kernel of $T$, is complemented in $E$ and $\text{im} T$, the image of $T$, is closed and complemented in $F$. 

Let $\mathcal{A}$ be a Banach algebra. A short, exact sequence $\Sigma : \{0\} \longrightarrow F \overset{\beta}{\longrightarrow} E \overset{\theta}{\longrightarrow} H \longrightarrow \{0\}$ of Banach left $\mathcal{A}$-modules is admissible if $\theta$ has a bounded right inverse. We say $\Sigma$ splits if $\theta$ has a bounded right inverse that is also a left module homomorphism, and $\Sigma$ approximately splits if there is a net $\rho_i : H \longrightarrow E$ of right inverse maps to $\theta$ such that

$$a . \rho_i(x) - \rho_i(a . x) \longrightarrow 0 \quad (a \in \mathcal{A}, \ x \in H).$$

**Definition 2.1.** Let $\mathcal{A}$ be a Banach algebra. A left Banach $\mathcal{A}$-module $P$ is approximately projective, if there is a net $\rho_i : P \longrightarrow \mathcal{A}^\# \hat{\otimes} P$ of right inverse maps to $\Pi_{\mathcal{A}^\#, P}$ such that

$$a . \rho_i(x) - \rho_i(a . x) \longrightarrow 0 \quad (a \in \mathcal{A}, \ x \in P).$$

**Proposition 2.2.** Let $\mathcal{A}$ be a Banach algebra. For a left Banach $\mathcal{A}$-module $P$, the following are equivalent:

(i) $P$ is approximately projective.

(ii) If $E$ and $F$ are left Banach $\mathcal{A}$-modules, if $\theta \in \mathcal{A} \mathcal{L}(E, F)$ is surjective and admissible, and if $\sigma \in \mathcal{A} \mathcal{L}(P, F)$, then there is a net $\rho_i : P \longrightarrow \mathcal{A}^\# \hat{\otimes} P$ such that $\sigma = \theta \rho_i$ for each $i$, and

$$a . \rho_i(x) - \rho_i(a . x) \longrightarrow 0 \quad (a \in \mathcal{A}, \ x \in P).$$

(iii) Every admissible, short exact sequence $\Sigma : 0 \longrightarrow F \overset{\beta}{\longrightarrow} E \overset{\theta}{\longrightarrow} P \longrightarrow 0$, of left Banach $\mathcal{A}$-modules, approximately splits.

**Proof.** (i) $\Longrightarrow$ (ii) Let $T \in \mathcal{L}(F, E)$ be a right inverse of $\theta$. Define $S := T \sigma \Pi_{\mathcal{A}^\#, P}$. By [Run, Exercise 5.1.1], there is $\beta \in \mathcal{A} \mathcal{L}(\mathcal{A}^\# \hat{\otimes} P, E)$ such that $\beta(e \otimes x) = Sx$, for each $x \in P$. Then for $a \in \mathcal{A}^\#$ and $x \in P$, we have

$$\theta \beta(a \otimes x) = \theta(a . \beta(e \otimes x)) = a . \theta(Sx) = a . \sigma \Pi_{\mathcal{A}^\#, P}(e \otimes x) = \sigma \Pi_{\mathcal{A}^\#, P}(a \otimes x)$$

so that $\sigma \Pi_{\mathcal{A}^\#, P} = \theta \beta$.

Let $\gamma_i : P \longrightarrow \mathcal{A}^\# \hat{\otimes} P$ be a net of right inverse maps to $\Pi_{\mathcal{A}^\#, P}$ such that

$$a . \gamma_i(x) - \gamma_i(a . x) \longrightarrow 0 \quad (a \in \mathcal{A}, \ x \in P).$$
Put $\rho_i := \beta \gamma_i$, so we have $\theta \rho_i = \sigma$ for each $i$, and
\[
\begin{align*}
    a \cdot \rho_i(x) - \rho_i(a \cdot x) &= a \cdot (\beta \gamma_i(x)) - \beta \gamma_i(a \cdot x) \\
    &= \beta(a \cdot \gamma_i(x)) - \beta(\gamma_i(a \cdot x)) \\
    &= \beta(a \cdot \gamma_i(x) - \gamma_i(a \cdot x)) \\
\end{align*}
\rightarrow 0 .
\]

(ii) $\Rightarrow$ (iii) Let $\Sigma : 0 \rightarrow F \xrightarrow{\beta} E \xrightarrow{\theta} P \rightarrow 0$ be an admissible, short exact sequence. By assumption, there is a net $\rho_i : P \rightarrow E$ of right inverse maps to $\theta$ such that
\[
    a \cdot \rho_i(x) - \rho_i(a \cdot x) \rightarrow 0 \quad (a \in A, \ x \in P) ,
\]
this means $\Sigma$ approximately splits.

(iii) $\Rightarrow$ (i) Apply (iii) to the short exact sequence
\[
    0 \rightarrow \ker \Pi_{A^\# , P} \rightarrow A^\# \otimes P \xrightarrow{\Pi_{A^\# , P}} P \rightarrow 0 .
\]

Let $A$ be a Banach algebra, and let $E$ be a Banach space. Then $\mathcal{L}(A, E)$ becomes a left Banach $A$-module by letting
\[
    (a \cdot T)(x) := T(ax) \quad (a, x \in A, \ T \in \mathcal{L}(A, E)) .
\]

If $E$ is also a left Banach $A$-module, there is a canonical left $A$-module homomorphism
\[
    \Pi^{A,E} : E \rightarrow \mathcal{L}(A, E) , \quad \Pi^{A,E}(a)(x) := a \cdot x \quad (a \in A, \ x \in E) .
\]

**Definition 2.3.** Let $A$ be a Banach algebra. A left Banach $A$-module $I$ is approximately injective, if there is a net $\rho_i : \mathcal{L}(A^\#, I) \rightarrow I$ of left inverse maps to $\Pi^{A^\#, I}$ such that
\[
    a \cdot \rho_i(T) - \rho_i(a \cdot T) \rightarrow 0 \quad (a \in A, \ T \in \mathcal{L}(A^\#, I)) .
\]

In this paper, we just consider left modules, but we can prepare analogues definitions of approximately injectivity for right modules and bimodules, and obtain similar results for them.
The proof of the following proposition is similar to Proposition 2.2.

**Proposition 2.4.** Let $A$ be a Banach algebra. For a left Banach $A$-module $I$, the following are equivalent:

(i) $I$ is approximately injective.

(ii) If $E$ and $F$ are left Banach $A$-modules, if $\theta \in _A \mathcal{L}(E,F)$ is injective and admissible, and if $\sigma \in _A \mathcal{L}(F,I)$, then there is a net $\rho_i : E \to I$ such that $\sigma = \rho_i \theta$ for each $i$, and

$$a \cdot \rho_i(x) - \rho_i(a \cdot x) \to 0 \quad (a \in A, \ x \in E).$$

(iii) Every admissible, short exact sequence $0 \to I \to E \to \frac{E}{T} \to 0$, of left Banach $A$-modules, approximately splits.

Let $A$ be a Banach algebra. A Banach $A$-module $E$ is **faithful** if $x \in E$ is such that $a \cdot x = 0$ for all $a \in A$, then $x = 0$.

**Definition 2.5.** Let $A$ be a Banach algebra. A left Banach $A$-module $E$ is **approximately faithful** if a net $(x_i)_i \subseteq E$ is such that $\lim_i a \cdot x_i = 0$, for each $a \in A$, then $\lim_i x_i = 0$.

Define the map $\Pi : E \to \mathcal{L}(A,E)$ by $\Pi(x) := \Pi^{A,E}(x) |_A$, for $x \in E$. Following [DP, Proposition 1.7], we have the next Proposition.

**Proposition 2.6.** Let $A$ be a Banach algebra, and let $E$ be an approximately faithful left Banach $A$-module. Then $E$ is approximately injective if and only if there is a net $\rho_i : \mathcal{L}(A,E) \to E$ of left inverse maps to $\Pi$ such that

$$a \cdot \rho_i(T) - \rho_i(a \cdot T) \to 0 \quad (a \in A, \ T \in \mathcal{L}(A,E)).$$

**Proof.** $\implies$ Suppose that $E$ is approximately injective. Then there is a net $\gamma_i : \mathcal{L}(A^\sharp,E) \to E$ of left inverse maps to $\Pi^{A,E}$ such that

$$a \cdot \gamma_i(S) - \gamma_i(a \cdot S) \to 0 \quad (a \in A, \ S \in \mathcal{L}(A^\sharp,E)).$$

Take $S \in \mathcal{L}(A^\sharp,E)$ such that $S |_A = 0$. Then $a \cdot S = 0$, for $a \in A$, and so $\gamma_i(a \cdot S) = 0$. Then $\lim_i a \cdot \gamma_i(S) = 0$. Since $E$ is approximately faithful, $\lim_i \gamma_i(S) = 0$. 

Now take $T \in \mathcal{L}(\mathcal{A}, E)$ and extend to $\tilde{T} \in \mathcal{L}(\mathcal{A}^\# \mathcal{A}, E)$ by $\tilde{T}(e_{\mathcal{A}^\#}) = 0$.

Setting $\rho_i(T) := \gamma_i(\tilde{T})$, for $T \in \mathcal{L}(\mathcal{A}, E)$, we have

$$a \cdot \rho_i(T) - \rho_i(a \cdot T) = a \cdot \gamma_i(\tilde{T}) - \gamma_i(a \cdot \tilde{T}) + \gamma_i(a \cdot \tilde{T} - a \cdot \tilde{T}) \rightarrow 0 \quad (a \in \mathcal{A}),$$

because $(a \cdot \tilde{T} - a \cdot \tilde{T}) |_{\mathcal{A}} = 0$. Also, it is clear that $\rho_i \Pi = id_E$, for each $i$.

$\Leftarrow$ Let $\gamma_i : \mathcal{L}(\mathcal{A}, E) \rightarrow E$ be a net of left inverse maps to $\Pi$ such that

$$a \cdot \gamma_i(T) - \gamma_i(a \cdot T) \rightarrow 0 \quad (a \in \mathcal{A}, T \in \mathcal{L}(\mathcal{A}, E)).$$

Define $\rho_i(T) := \gamma_i(T |_{\mathcal{A}})$. Then for $a \in \mathcal{A}$ and $T \in \mathcal{L}(\mathcal{A}^\# \mathcal{A}, E)$, we have

$$a \cdot \rho_i(T) - \rho_i(a \cdot T) = a \cdot \gamma_i(T |_{\mathcal{A}}) - \gamma_i((a \cdot T) |_{\mathcal{A}}) = a \cdot \gamma_i(T |_{\mathcal{A}}) - \gamma_i(a \cdot (T |_{\mathcal{A}})) \rightarrow 0,$$

and $\rho_i \Pi^{\mathcal{A}^\# \mathcal{A}} = id_E$, for each $i$.

Proposition 2.7. Let $\mathcal{A}$ be a Banach algebra, and let $\mathcal{A}$ be approximately injective as a left Banach $\mathcal{A}$-module. Then $\mathcal{A}$ has an approximate right identity.

Proof. There is a net $\rho_i : \mathcal{L}(\mathcal{A}^\#, \mathcal{A}) \rightarrow \mathcal{A}$ of left inverse maps to $\Pi^{\mathcal{A}^\# \mathcal{A}}$ such that

$$a \cdot \rho_i(T) - \rho_i(a \cdot T) \rightarrow 0 \quad (a \in \mathcal{A}, T \in \mathcal{L}(\mathcal{A}^\# \mathcal{A})),$$

where $\Pi^{\mathcal{A}^\# \mathcal{A}}(a)(b) = ba$, for $a \in \mathcal{A}$ and $b \in \mathcal{A}^\#$.

Let $P \in \mathcal{L}(\mathcal{A}^\#, \mathcal{A})$ be the natural projection of $\mathcal{A}^\#$ onto $\mathcal{A}$, so that $P |_{\mathcal{A}} = id_{\mathcal{A}}$, and $a \cdot P = \Pi^{\mathcal{A}^\# \mathcal{A}}(a)$, for $a \in \mathcal{A}$. Set $u_i := \rho_i(P)$. Then

$$au_i - a = a \rho_i(P) - \rho_i \Pi^{\mathcal{A}^\# \mathcal{A}}(a) = a \rho_i(P) - \rho_i(a \cdot P) \rightarrow 0,$$

so that $(u_i)$ is an approximate right identity for $\mathcal{A}$. \qed

Corollary 2.8. Let $\mathcal{A}$ be a Banach algebra which has a left identity. Suppose that $\mathcal{A}$ is approximately injective as a left Banach $\mathcal{A}$-module. Then $\mathcal{A}$ has an identity.

Proof. Let $e$ be a left identity for $\mathcal{A}$. By Proposition 2.7, $\mathcal{A}$ has an approximate right identity, say $(u_i)_i$. Since $u_i \rightarrow e$, for each $a \in \mathcal{A}$, $au_i \rightarrow ae$, and $au_i \rightarrow a$, so that $ae = a$. Therefore $e$ is an identity for $\mathcal{A}$. \qed
3 Approximate amenability

Let $\mathcal{A}$ be an approximately amenable Banach algebra. By [GhLZ, Theorem 2.1] and [GhL, Theorem 2.2], every admissible, short exact sequence, approximately splits, so that, by Proposition 0.4, every left Banach $\mathcal{A}$-module is approximately injective.

Our reference for amenability of locally compact groups is [P]. Let $G$ be a locally compact group and let $E$ be a Banach left $L^1(G)$-module, where $L^1(G)$ is the group algebra of $G$. Let $G$ be amenable, then $L^1(G)$ is approximately amenable, [GhL, Theorem 3.2]. Therefore by above paragraph, $E$ is approximately injective. In this section we study a form of converse to this result. We keep the notations of [DP].

Let $G$ be a locally compact group and let $E$ be a Banach left $L^1(G)$-module. An element $\lambda \in E^*$ is an augmentation-invariant functional, if

$$\langle f . x, \lambda \rangle = \varphi_G(f) \langle x, \lambda \rangle \quad (f \in L^1(G), \ x \in E),$$

where $\varphi_G : M(G) \to \mathbb{C}$, $\mu \mapsto \mu(G)$, is the augmentation character on $M(G)$. The module $E$ is augmentation-invariant if there is a non-zero augmentation functional on $E^*$, see [DP] for details.

Let $\mathcal{A} = L^1(G)$. For $s \in G$, $T \in \mathcal{L}(\mathcal{A}, E)$ and $\Lambda \in \mathcal{L}(\mathcal{A}, E)^*$, we recall the definitions of the right-translate $T . s$ and the left-translate $s . \Lambda$:

$$(T . s)(f) := T(L_s f) \quad \text{and} \quad (T . s . \Lambda) = (T . s, \Lambda) \quad (f \in \mathcal{A}),$$

where, $L_s f$ is the complex map from $G$, given by $x \mapsto f(sx)$. In this case we have $(T . s) . f = T . L_s f$, for every $f \in \mathcal{A}$.

For $f \in \mathcal{A}$, define $f^\circ := \bar{f}^*$, where $f^*$ is the involution of $f$. It is easy to see that

$$f^{\circ \circ} = f \ , \ (f * g)^\circ = g^\circ * f^\circ \ , \ \varphi_G(f^\circ) = \varphi_G(f) \quad (f \in \mathcal{A}).$$

For $T \in \mathcal{L}(\mathcal{A}, E)$, define $T^\circ \in \mathcal{L}(\mathcal{A}, E)$ by $T^\circ(f) = T(f^\circ)$. Then for $f \in \mathcal{A}$,

$$(T . f)^\circ = f^\circ . T^\circ.$$
Let $\mathcal{A} = L^1(G)$ and let $E$ be an augmentation-invariant left Banach $\mathcal{A}$-module. Take a non-zero, augmentation functional $\lambda_0 \in E^*$, and take $x_0 \in E$ with $\langle x_0, \lambda_0 \rangle = 1$. Setting $T_0 = \Pi(x_0) \in \mathcal{L}(\mathcal{A}, E)$, we may suppose that $\|T_0\| = \|\tilde{T}_0\| = 1$. We keep these notations in the following lemmas. The following lemmas are similar to [DP, Lemmas 4.3, 4.4, 4.5].

**Lemma 3.1.** Let $E$ be an augmentation-invariant left Banach $L^1(G)$-module. Let $E$ be approximately faithful and approximately injective. Then there is a net $(\Lambda_i) \subseteq \mathcal{L}(\mathcal{A}, E)^*$ such that $\langle T_0, \Lambda_i \rangle = 1$, for all $i$ and $L_s \Lambda_i - \Lambda_i \rightharpoonup 0$, for every $s \in G$.

**Proof.** By Proposition 2.6, there is a net $\rho_i : \mathcal{L}(\mathcal{A}, E) \rightarrow E$ of left inverse maps to $\Pi$, such that

$$a . \rho_i(T) - \rho_i(a . T) \longrightarrow 0 \quad (a \in \mathcal{A}, T \in \mathcal{L}(\mathcal{A}, E)) .$$

In particular $\rho_i(T_0) = \rho_i \Pi(x_0) = x_0$, for each $i$. Also for $f \in \mathcal{A}$ and $T \in \mathcal{L}(\mathcal{A}, E)$, we have

$$(\lambda_0 \rho_i)(f^\circ . T^\circ) - \varphi_G(f)(\lambda_0 \rho_i)(T^\circ) = \lambda_0(\rho_i(f^\circ . T^\circ) - f^\circ . \rho_i(T^\circ)) \longrightarrow 0 .$$

Define $\Lambda_i : \mathcal{L}(\mathcal{A}, E) \rightarrow \mathbb{C}$, by $\langle T, \Lambda_i \rangle := \langle T^\circ, \lambda_0 \rho_i \rangle$. Then $\Lambda_i \in \mathcal{L}(\mathcal{A}, E)^*$ and $\langle T_0^\circ, \Lambda_i \rangle = 1$.

Take $f \in \mathcal{A}$ such that $\varphi_G(f) = 1$. Then for $T \in \mathcal{L}(\mathcal{A}, E)$

$$\langle T . f, \Lambda_i \rangle - \langle T, \Lambda_i \rangle = \langle f^\circ . T^\circ, \lambda_0 \rho_i \rangle - \langle T^\circ, \lambda_0 \rho_i \rangle \rightarrow 0 .$$

Let $s \in G$, let $f \in \mathcal{A}$, and let $g := (L_s f)^\circ$, then we obtain

$$\langle T, L_s(\Lambda_i) \rangle - (g . T^\circ, \lambda_0 \rho_i) = \langle T . s, \Lambda_i \rangle - \langle (T . s) . f, \Lambda_i \rangle \rightarrow 0 .$$

Thus

$$\langle T, L_s(\Lambda_i) \rangle - \langle \rho_i(T^\circ), \lambda_0 \rangle = \langle T, L_s(\Lambda_i) \rangle - (g . T^\circ, \lambda_0 \rho_i) + ((\rho_i(g . T^\circ), \lambda_0) - \langle \rho_i(T^\circ), \lambda_0 \rangle)
= (\langle T, L_s(\Lambda_i) \rangle - \langle g.T^\circ, \lambda_0 \rho_i \rangle) + \langle \rho_i(g . T^\circ) - g . \rho_i(T^\circ), \lambda_0 \rangle
\rightarrow 0 .$$
Finally,
\[
\langle T, L_s(\Lambda_i) \rangle - \langle T, \Lambda_i \rangle = \langle T, L_s(\Lambda_i) \rangle - \langle \rho_i(T^c), \lambda_0 \rangle + \langle \rho_i(T^c), \lambda_0 \rangle - \langle T, \Lambda_i \rangle \longrightarrow 0 ,
\]
so that \( L_s \Lambda_i - \Lambda_i \xrightarrow{w^*} 0 \), for every \( s \in G \).

Let \( E \) be a dual left Banach \( A \)-module, say \( E = F^* \), for a right \( A \)-module \( F \). Then \( \mathcal{L}(A, E) \cong (A \hat{\otimes} F)^* \) as a Banach space. Let \( X := A \hat{\otimes} F \), and define \( L_s(f \otimes y) = L_s f \otimes y \), for \( f \in A \) and \( y \in F \), so that
\[
\langle T, L_s x \rangle = \langle T . s, x \rangle \quad (x \in X, \ T \in \mathcal{L}(A, E)) .
\]

**Lemma 3.2.** Let \( E \) be an augmentation-invariant left Banach \( L^1(G) \)-module, let \( E \) be dual module, and let \( E \) be approximately faithful and approximately injective. Then there is a net \( (v_\alpha) \subseteq X \) such that \( \langle T_0^c, v_\alpha \rangle = 1 \), for each \( \alpha \), and \( ||L_s v_\alpha - v_\alpha|| \longrightarrow 0 \), for every \( s \in G \).

**Proof.** Let \((\Lambda_i)_{i \in I}\) be as Lemma 3.1. For finite sets \( \{s_1, ..., s_k\} \subseteq G \), with \( s_1 = e \), \( \{T_1, ..., T_l\} \subseteq \mathcal{L}(A, E) \), and \( F \subseteq I \), set \( \alpha := (\{s_1, ..., s_k\}, \{T_1, ..., T_l\}, F) \).

For each \( \alpha \), choose \( u_\alpha \in X \) such that
\[
\langle T_0^c, u_\alpha \rangle = 1 \quad \text{and} \quad \langle T_m . s_n, u_\alpha \rangle = \langle T_m . s_n, \Lambda_j \rangle ,
\]
for \( m = 1, ..., l \), \( n = 1, ..., k \), and \( j = \max \{i : i \in F\} \).

Then, for \( s \in G \) and \( T \in \mathcal{L}(A, E) \), we have
\[
\langle T, L_s u_\alpha \rangle - \langle T, u_\alpha \rangle = \langle T . s, \Lambda_j \rangle - \langle T, \Lambda_j \rangle = \langle T, L_s \Lambda_j \rangle - \langle T, \Lambda_j \rangle \longrightarrow 0 ,
\]
so that \( L_s u_\alpha - u_\alpha \xrightarrow{w^*} 0 \) in \( X \).

Let \( \{s_1, ..., s_k\} \) be a finite subset of \( G \). Consider the Banach space \( Y = \bigoplus_{n=1}^{k} X_n \), where \( X_n = X \) for \( n = 1, ..., k \). Define a linear operator
\[
W : X \longrightarrow Y , \ x \longmapsto (L_{s_1} x - x, ..., L_{s_k} x - x) .
\]

The set \( C = \{x \in X : \langle T_0^c, x \rangle = 1\} \) is convex in \( X \), and so \( W(C) \) is convex in \( Y \).
By the above argument and Mazur’s Theorem, it follows that 0 belongs to the \( \| \cdot \| \)-closure \( W(C) \) in \( Y \). The existence of the required net \( (v_n) \) follows.

We write \( P(G) \) for the set of all positive functions \( f \in L^1(G) \) such that \( \|f\|_1 = 1 \).

**Lemma 3.3.** With the assumptions of Lemma 3.2, there is a net \( (h_\alpha) \subseteq P(G) \) such that \( \|L_s h_\alpha - h_\alpha\|_1 \to 0 \), for each \( s \in G \).

**Proof.** Using Lemma 3.2, it is similar to [DP, Lemma 4.5].

**Theorem 3.4.** Let \( E \) be a dual left Banach \( L^1(G) \)-module. Let \( E \) be approximately faithful and augmentation-invariant. Then \( E \) is approximately injective if and only if \( G \) is amenable.

**Proof.** Suppose that \( E \) is approximately injective. Therefore there is a net \( (h_\alpha)_\alpha \) in \( P(G) \) as specified in Lemma 3.3. Then, by [P, Proposition 0.8], \( G \) is amenable.

Conversely, as we mentioned earlier, the amenability of \( G \) implies the approximately injectivity of \( E \).

An analogous result holds if we exchange ’right’ and ’left’ in the above theorem.

Let \( A \) be a Banach algebra. A Banach left \( A \)-module \( E \) is called approximately flat if the dual module \( E^* \) is approximately injective as a Banach right \( A \)-module.

For a locally compact group \( G \), the spaces \( \ell^1(G) \), \( \ell^\infty(G) \) and \( c_0(G) \) are standard, and we have \( c_0(G)^* = \ell^1(G) \) and \( \ell^1(G)^* = \ell^\infty(G) \).

**Corollary 3.5.** Let \( G \) be a discrete group. Then the following are equivalent:

(i) \( G \) is amenable.

(ii) \( \ell^1(G) \) is approximately injective as a left Banach \( \ell^1(G) \)-module.

(iii) \( \ell^\infty(G) \) is approximately flat as a left Banach \( \ell^1(G) \)-module.

(iv) \( c_0(G) \) is approximately flat as a left Banach \( \ell^1(G) \)-module.

**Proof.** By [DP, Example 4.2], \( \ell^1(G) \) and \( \ell^\infty(G)^* \) are augmentation-invariant. Because \( \ell^1(G) \) is unital, \( \ell^1(G) \) and \( \ell^\infty(G)^* \) are also approximately faithful as left Banach \( \ell^1(G) \)-module. Therefore the equivalences \( (i) \iff (ii) \iff (iii) \) follow from Theorem 3.4. Since \( \ell^1(G) = c_0(G)^* \) is approximately injective as a right Banach \( \ell^1(G) \)-module if and only if \( G \) is amenable, we also have \( (i) \iff (iv) \).

\( \square \)
Acknowledgment

The main results of this paper were obtained while author was at the University of Manitoba. The author is grateful to the Azad University for its support and also to the University of Manitoba for hospitality.

References


